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Abstract

In the present paper, we characterize the topological types of the exteriors of tunnel number two knots in the 3-sphere S^3 with weakly reducible genus three Heegaard splittings.

1. Introduction

Let K be a knot in the 3-sphere S^3, $N(K)$ a regular neighborhood of K in S^3 and $E(K)=cl(S^3-N(K))$ the exterior of K. Then the tunnel number of K, denoted by $t(K)$, is defined as the minimal number of mutually disjoint arcs $\gamma_1, \gamma_2, \cdots, \gamma_t$ properly embedded in $E(K)$ such that $cl(E(K)-N(\partial E(K)\cup\gamma_1\cup\cdots\cup\gamma_t))$ is a handlebody, where $N(\partial E(K)\cup\gamma_1\cup\cdots\cup\gamma_t)$ is a regular neighborhood of $\partial E(K)\cup\gamma_1\cup\cdots\cup\gamma_t$ in $E(K)$. We call the collection of the arcs $\{\gamma_1, \gamma_2, \cdots, \gamma_t\}$ an unknotting tunnel system of K.

Put $H_1=N(\partial E(K)\cup\gamma_1\cup\cdots\cup\gamma_t)$, and $H_2=cl(E(K)-H_1)$. Then $E(K)=H_1\cup H_2$ is a genus $t(K)+1$ Heegaard splitting of $E(K)$, where H_1 is a compression body and H_2 is a handlebody. We say that the Heegaard splitting (H_1, H_2) is weakly reducible if there is an essential disk, say D_i, properly embedde in H_i $(i=1,2)$ such that $D_1\cap D_2=\emptyset$, and that (H_1, H_2) is strongly irreducible if it is not weakly reducible. For the definition of compression body, we refer [CG], and the notion of weak reducibility and strong irreducibility of Heegaard splittings is also due to [CG].

Let V be a solid torus and K a knot in $intV$. Let $N_V(K)$ be a regular neighborhood of K in V and $E_V(K)=cl(V-N_V(K))$ the exterior. We say that K is a tunnel number one knot of type I in V if there is an arc γ properly embedde in $E_V(K)$ such that $\gamma\cap\partial N_V(K)=\partial\gamma$ and $cl(E_V(K)-N(\partial N_V(K)\cup\gamma))$ is a genus two compression body, where $N(\partial N_V(K)\cup\gamma)$ is a regular neighborhood of $\partial N_V(K)\cup\gamma$ in $E_V(K)$, and we say that K is a tunnel number one knot of type II in V if there is an arc γ properly embedded in $E_V(K)$ such that γ connects a point in $\partial N_V(K)$ and a point in ∂V and $cl(E_V(K)-N(\partial N_V(K)\cup\gamma\cup\partial V))$ is a genus two handlebody, where $N(\partial N_V(K)\cup\gamma\cup\partial V)$ is a regular neighborhood of $\partial N_V(K)\cup\gamma\cup\partial V$ in $E_V(K)$. In this definition, we call the arc γ an unknotting tunnel of K. Then in the present paper we show :

Theorem 1.1 *Let K be a tunnel number two knot in S^3. Suppose a genus three Heegaard splitting of $E(K)$ is weakly reducible, then $E(K)$ is obtained from $E(K_1)$ and $E_V(K_2)$ by glueing $\partial E(K_1)$ and ∂V, where K_1 is some tunnel number one knot in S^3 and K_2 is some tunnel number one knot in a solid torus V.*

In this theorem, since the glueing torus $\partial E(K_1) = \partial V$ is essential in $E(K)$, we have :

Corollary 1.2 *Let K be a tunnel number two knot in S^3. If $E(K)$ has a weakly reducible genus three Heegaard splitting, then $E(K)$ contains an essential torus which divides $E(K)$ into a tunnel number one knot exterior in S^3 and a tunnel number one knot exterior in a solid torus.*

In [M1], we characterized the knot types of composite tunnel number two knots in S^3, i.e., if a composite knot $K_1 \# K_2$ has tunnel number two, then (1) $t(K_1) = t(K_2) = 1$ and at least one of K_1 and K_2 has a $(1, 1)$-decomposition, or (2) at least one of K_1 and K_2, say K_1, is a 2-bridge knot and K_2 has a 2-string essential free tangle decomposition such that at least one of the two tangles has an unknotted component. Hence we show :

Corollary 1.3 *Let K_1, K_2 be two knots in S^3 with $t(K_1)+t(K_2)>2$. Suppose $t(K_1 \# K_2) = 2$, then any genus three Heegaard splitting of $E(K_1 \# K_2)$ is strongly irreducible.*

Remark 1 As the converse of Theorem 1.1, we see, in the proof of Theorem 1.1, that the union of a tunnel number one knot exterior in S^3 and a tunnel number one knot exterior in a solid torus is a tunnel number two knot exterior in S^3 by some suitable glueing map, and that any genus three Heegaard splitting of such a tunnel number two knot exterior is weakly reducible.

Concerning relation between tunnel number one knots in S^3 and tunnel number one knots in a solid torus, we show :

Proposition 1.4 *Tunnel number one knots in a solid torus can be regarded as tunnel number one knots in S^3, and vice versa.*

Thrughout the present paper, we will work in the piecewise linear category. For a manifold X and a subcomplex Y in X, we denote a regular neighborhood of Y in X by $N(Y,X)$ or $N(Y)$ simply.

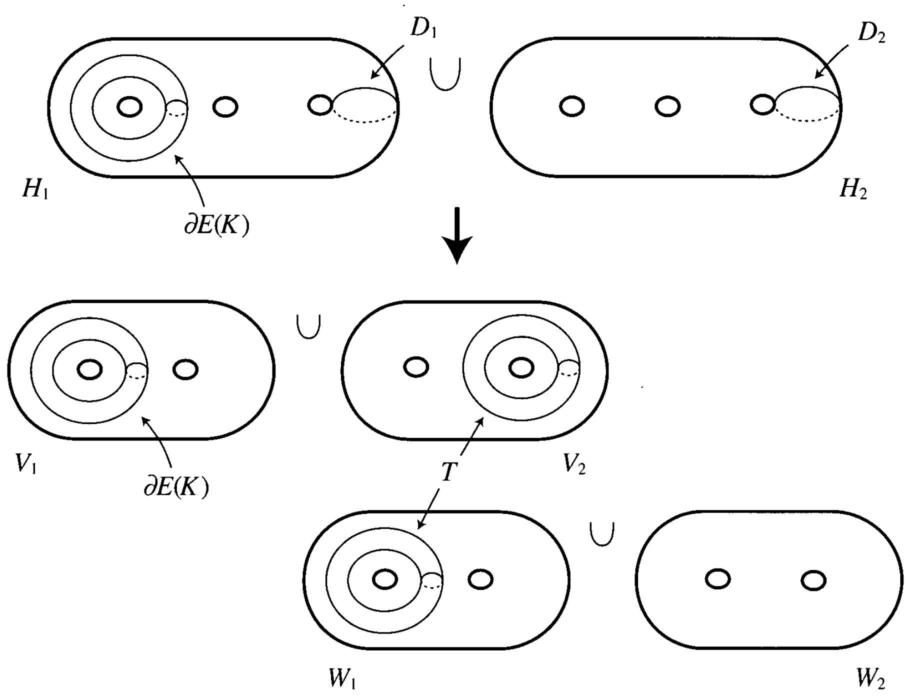

Figure 1

2. Proof of Theorem 1.1

Let $E(K) = H_1 \cup H_2$ be a weakly reducible genus three Heegaard splitting with $\partial\_H_1 = \partial E(K)$, and $D_1 \subset H_1$ and $D_2 \subset H_2$ be essential disks with $D_1 \cap D_2 = \emptyset$. Then we have the following three cases.

Case I : Both D_1 and D_2 are non-separating in H_1 and in H_2 respectively.

Put $H_1' = cl(H_1 - N(D_1))$, $H_2' = cl(H_2 - N(D_2))$, and put $V_1 = cl(H_1' - N(\partial H_1' - \partial E(K)))$, $V_2 = N(\partial H_1' - \partial E(K)) \cup N(D_2)$, $W_1 = N(\partial H_2') \cup N(D_1)$ and $W_2 = cl(H_2' - N(\partial H_2'))$ as illustrated in Figure 1.

Put $T = V_2 \cap W_1$. Then by the solid torus theorem, T is a boundary of a solid torus in $S^3 = E(K) \cup N(K)$, and $N(K)$ is contained in the sorid torus. Hence $W_1 \cup W_2$ is a knot exterior of some tunnel number one knot in S^3 because (W_1, W_2) is a genus two Heegaard splitting. And $V_1 \cup V_2$ is a knot extrior of some tunnel number one knot of type I in the solid torus bounded by T because (V_1, V_2) is a genus two Heegaard splitting.

Case II : Both D_1 and D_2 are separating in H_1 and in H_2 respectively. Let P_i be the torus with one hole bounded by ∂D_i in ∂H_i $(i = 1, 2)$. If $P_1 \cap P_2 \neq \emptyset$, then since $\partial D_1 \cap \partial D_2 = \emptyset$, we have $P_1 \subset P_2$ or $P_2 \subset P_1$. Then by some isotopy, we may assume that $P_1 = P_2$ and $\partial D_1 = \partial D_2$. Then $D_1 \cup D_2$ is a 2-sphere which bounds a 3-ball in $E(K)$. Then the knot K is a trivial knot or a tunnel number one knot, a contradiction.

Hence $P_1 \cap P_2 = \emptyset$. Let $T_i = P_i \cup D_i$ be a torus in H_i $(i = 1, 2)$. If T_1 bounds a solid torus in H_1, then we can take a meridian disk in the solid torus. Moreover, we can take a meridian disk in the solid torus bounded by T_2 in H_2, Then this case is reduced to Case I.

Suppose T_1 bounds a torus $\times I$ in H_1, say X, and T_2 bounds a solid torus in H_2, say Y. Put $H_1' = cl(H_1 - X)$, $H_2' = cl(H_2 - Y)$, and put $V_1 = cl(H_2' - N(\partial H_2'))$, $V_2 = N(\partial H_2') \cup X$, $W_1 = N(\partial H_1') \cup Y$ and $W_2 = cl(H_1' - N(\partial H_1))$ as illustrated in Figure 2.

Then by the reason similar to the proof of Case I, we see that $W_1 \cup W_2$ is a tunnel number one knot exterior in S^3, and $V_1 \cup V_2$ is a tunnel number one knot exterior of type II in a solid torus.

Case III : One of D_1 and D_2 is separating and the other is non-separating.

Suppose D_1 is separating in H_1 and D_2 is non-separating in H_2. Since $\partial D_1 \cap \partial D_2 = \emptyset$, we can take a loop ℓ in $\partial H_1 = \partial H_2$ such that $\ell \cap \partial D_1 = \emptyset$ and $\ell \cap \partial D_2$ is a single point. Take a regular neighborhood of $D_2 \cup \ell$ in H_2, then it is a solid torus in H_2 and let D_2' be the frontier of the solid torus in H_2. Then D_2' is a separating essential disk in H_2 with $\partial D_1 \cap \partial D_2' = \emptyset$. Next suppose D_1 is non-separating in H_1 and D_2 is separating in H_2. Then similarly as above, we can take a separating disk D_1' in H_1 with $\partial D_1' \cap \partial D_2 = \emptyset$. Hence Case III is reduced to Case II, and this completes the proof of Theorem 1.1. □

3. Proof of Corollary 1.3

By [M1], we may assume that K_1 is a 2-bridge knot and K_2 has a 2-string essential free

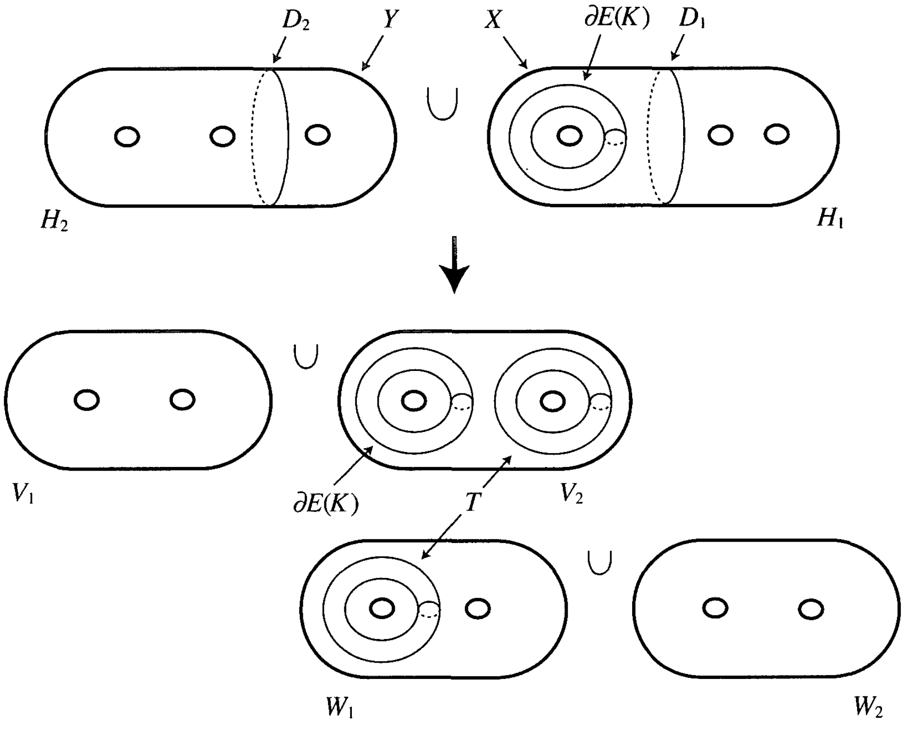

Figure 2

tangle decomposition such that at least one of the two tangles has an unknotted component. Put $K=K_1\#K_2$ and suppose $E(K)$ has a genus three weakly reducible Heegaard splitting. Then by Corollary 1.2, there is an essential torus, say T, in $E(K)$ which divides $E(K)$ into a tunnel number one knot exterior in S^3, say $E(K_1')$, and a tunnel number one knot exterior in a solid torus V, say $E_V(K_2')$.

Suppose T is a swallow follow torus in the connected sum. Then, since $t(K_1)=1$ and $t(K_2)=2$, $E(K_1)$ is homeomorphic to $E(K_1')$ and $E(K_2)$ is homeomorphic to $E_V(K_2')\cup V'$ for some solid torus V'. This shows that $E(K_2)$ has a genus two Heegaard splitting and $t(K_2)=1$, a contradiction. Hence T is not a swallow follow torus.

Let A be the decomposing annulus properly embedded in $E(K)$ corresponding to the connected sum of K.

First suppose $T\cap A=\emptyset$. Then we have the following two cases.

Case I : $T\subset E(K_1)$. Since T is not a swallow follow torus, T is an essential torus in $E(K_1)$. But 2-bridge knot exterior contains no essential torus by [Sc], a contradiction.

Case II : $T\subset E(K_2)$. By the same reason as that in Case I, T is an essential torus in $E(K_2)$. But by [Oz, Theorem 1.2 and Lemma 1.3] or by [M2, Proposition 2.1], this is a contradiction.

Hence $T\cap A\neq\emptyset$. Then, since we may assume that each component of $T\cap A$ is an essential loop in both T and A, we can take an essential annulus properly embedded in the 2-bridge knot exterior $E(K_1)$ whose boundary components are meridian loops. But this is a contradiction because 2-bridge knots are prime. Thus $E(K)$ has no genus three weakly reducible Heegaard splitting, and this completes the proof of Corollary 1.3. □

4. Proof of Proposition 1.4

Let K be a tunnel number one knot in a solid torus V and γ an unknotting tunnel of K. Then, put $V_1=N(K\cup\gamma)$ and $V_2=cl(V-V_1)$ if K is of type I, and put $V_1=N(K\cup\gamma\cup\partial V)$ and $V_2=cl(V-V_1)$ if K is of type II. Then (V_1,V_2) is a genus two Heegaard splitting of V as illustrated in Figure 3.

Let V' be a solid torus and gule ∂V and $\partial V'$ with some glueing map for $V\cup V'$ to be S^3. Put $H_1=V_1$ and $H_2=V_2\cup V'$ if K is of type I, and put $H_1=V_1\cup V'$ and $H_2=V_2$ if K is of type II. Then (H_1,H_2) is a genus two Heegaard splitting of S^3, and this shows that tunnel number one knots in a solid torus correspond to tunnel numer one knots in S^3. Moreover, suppose K is of type II, and let D_1 be a meridian disk in V_1 which intersects K in a single point. Take $N(D_1)$ in V_1 and put $H_1'=cl(H_1-N(D_1))$, $H_2'=H_2\cup N(D_1)$. Then, since H_2' is homeomorphic to $cl(S^3-V')$, H_2' is a solid torus and $N(D_1)\cap K$ is a trivial arc in the solid torus. Hence $(H_1',H_1'\cap K)\cup(H_2',H_2'\cap K)$ is a $(1,1)$-decomposition of (S^3,K), and K is a $(1,1)$ - knot in $S^3=V\cup V'$.

Conversely, let K be a tunnel number one knot in S^3, and (H_1,H_2) a genus two Heegaard splitting of S^3 which contains K as a core of a handle of H_1. Then, by [Wa], there is a

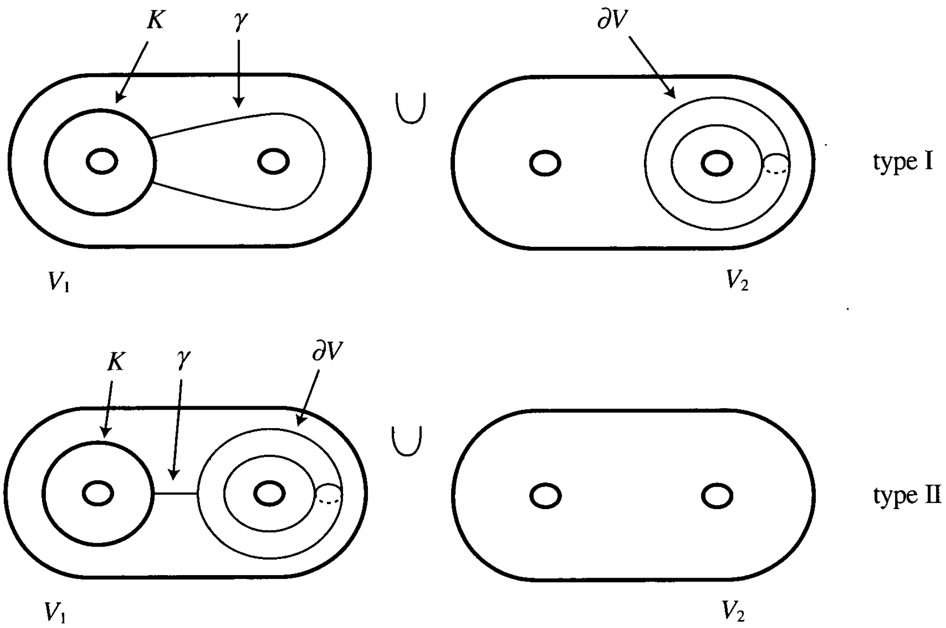

Figure 3

meridian disk, say D_2, in H_2 which makes a cancelling pair together with some meridian disk in H_1. Hence, put $V_1 = cl(H_1 - N(\partial H_1))$ and $V_2 = N(\partial H_1) \cup N(D_2)$, then (V_1, V_2) is a genus two Heegaard splitting of the solid torus $H_1 \cup N(D_2)$ and this shows that K is a tunnel number one knot of type I in the solid torus $H_1 \cup N(D_2)$. Moreover, if K has a $(1, 1)$-decomposition, then by tracing back the above arguments, we see that K can be regarded as a tunnel number one knot of type II in a solid torus. This completes the proof of Proposition 1.4. □

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